Lebesgue-type decompositions for some classes of linear relations and Radon-Nikodym derivatives

Seppo Hassi (University of Vaasa, Finland)

In this talk operator theoretic analogs and extensions of the Lebesgue decomposition for a pair of finite (or σ-finite) positive measures λ, μ on a measurable space (Ω, Σ), where Σ is a σ-algebra on the set Ω, are presented. The classical Lebesgue decomposition theorem states that λ can be decomposed with respect to μ: there exist (σ-)finite positive measures λ_r (the absolutely continuous part) and λ_s (the singular part), such that

$$\lambda = \lambda_r + \lambda_s, \quad \text{where} \quad \lambda_r \ll \mu, \quad \lambda_s \perp \mu. \tag{1}$$

Moreover, the absolutely continuous part λ_r admits the representation

$$\lambda_r(E) = \int_E f \, d\mu, \quad E \in \Sigma,$$

where $f \in L^1(\mu)$ is the Radon-Nikodym derivative.

In this talk we start by considering linear operators and, more generally, linear relations T (multivalued operators) from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ (i.e. the graph of T is a linear subspace of $\mathfrak{H} \times \mathfrak{K}$), and introduce Lebesgue-type decompositions of T as the sum

$$T = T_1 + T_2, \text{ where } T_1 \text{ is a closable operator and } T_2 \text{ is singular.} \tag{2}$$

In this general setting, the decomposition (2) need not be unique. On the other hand, one obtains a unique decomposition of the form (2), when T_1 is taken to be the maximal closable part (in the sense of domination), called the regular part T_{reg} of T, yielding to what we call the Lebesgue decomposition of T:

$$T = T_{\text{reg}} + T_{\text{sing}}. \tag{3}$$

This decomposition leads to the Lebesgue decomposition for lower semibounded forms $\mathfrak{t}$ established by B. Simon in 1978 (rewritten in our notation):

$$\mathfrak{t} = \mathfrak{t}_{\text{reg}} + \mathfrak{t}_{\text{sing}}. \tag{4}$$

Here the form $\mathfrak{t}_{\text{reg}}$ is closable, the form $\mathfrak{t}_{\text{sing}} \geq 0$ is singular, and $\mathfrak{t}_{\text{reg}}$ admits the following maximality property: $\mathfrak{t}_{\text{reg}} \geq \mathfrak{t}_1$ for all closable forms $\mathfrak{t}_1$ with $\mathfrak{t}_1 \leq \mathfrak{t}$. The connection between (3) and (4) is easily established by means of the notion of representing maps for semibounded forms, which we have studied recently together with Henk de Snoo. It can be also shown that the decompositions of the form (2) lead to the similar decompositions for the form $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$, where $\mathfrak{t}_1$ is closable and $\mathfrak{t}_2$ is singular. The decompositions also imply similar decompositions for a pair of forms $\mathfrak{t}$ and $\mathfrak{w}$, where $\mathfrak{t}_1$ is closable w.r.t. the form $\mathfrak{w}$ and $\mathfrak{t}_2$ is singular w.r.t. $\mathfrak{w}$. In particular, in this case the Lebesgue decomposition of the measures in (1) corresponds to the Lebesgue decomposition of the form (4), where $\mathfrak{t}_{\text{reg}}$ and $\mathfrak{t}_{\text{sing}}$ are then closable and singular w.r.t. the form $\mathfrak{w}$.

By specializing the decompositions (2) and (3) to the case, where the linear relation T (that is, its graph) is an operator range in the Cartesian product space $\mathfrak{H} \times \mathfrak{K}$, leads to Lebesgue-type decompositions for pairs of bounded operators $\Phi \in \mathbf{B}(\mathfrak{E}, \mathfrak{H})$ and $\Psi \in \mathbf{B}(\mathfrak{E}, \mathfrak{K})$, where $\mathfrak{E}$, $\mathfrak{H}$, and $\mathfrak{K}$ are Hilbert spaces. In this setting there is also natural operator theoretic analog for the Radon-Nikodym derivative. By considering Φ and Ψ as representing maps for the bounded forms

$$(A\eta, \zeta) = (\Phi\eta, \Phi\zeta), \quad (B\eta, \zeta) = (\Psi\eta, \Psi\zeta), \quad \eta, \zeta \in \mathfrak{E}, \tag{5}$$

one arrives at the Lebesgue-type decompositions for pairs of bounded positive operators $A, B \in \mathbf{B}(\mathfrak{E})$ studied originally by T. Ando in 1976.

The aim of this talk is to give an overview of our joint work and some recent papers with Henk de Snoo on Lebesgue-type decompositions (2) and (3), and their applicability in various other settings.